ALGEBRAIC CURVES
EXERCISE SHEET 4

Unless otherwise specified, k is an algebraically closed field.

Exercise 4.1. Show that all local rings of the affine line $\mathbb{A}^1_k$ are isomorphic to the same ring R.

Solution 1. Geometrically, this should be clear: the affine line $\mathbb{A}^1_k$ looks everywhere the same. More concretely, for any two points $a, b \in \mathbb{A}^1_k$, the translation

$$\tau_{b-a}\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$$
$$t \mapsto t + b - a$$

is an isomorphism (with inverse τ_{a-b}) with $\tau_{b-a}(a) = b$. In particular, as an isomorphism of algebraic varieties induces isomorphisms of local rings, the local rings at a and b are isomorphic.

Remark. To see that an isomorphism of affine varieties induces and isomorphism of local rings, you can use the following

- For any morphism of algebraic varieties $\varphi\colon U \to V$ and any point $p \in U$ there exists an induced map $\widetilde{\varphi}_p\colon \mathcal{O}_{V,\varphi(p)} \to \mathcal{O}_{U,p}$ of local rings.
- This is functorial: for $\varphi\colon U \to V$ and $\psi\colon V \to W$ and $p \in U$ we have
$$\widetilde{(\psi \circ \varphi)}_p = \widetilde{\varphi}_p \circ \widetilde{\psi}_{\varphi(p)}.$$
- In particular, if φ is an isomorphism, then $\widetilde{\varphi}_p$ is an isomorphism for all $p \in U$.

Exercise 4.2. An *affine algebraic group* is an affine variety G, whose underlying set is a group, such that the morphisms $i : G \to G,\ g \mapsto g^{-1}$ and $m : G \times G \to G,\ (g, h) \mapsto gh$ are polynomial maps. Let $V_1 = \mathbb{A}^1_k - \{0\}$ and $V_2 = V(xy - 1)$. From the first exercise, we call R the local ring of $\mathbb{A}^1_k$ at any point.

(1) Show that $\mathcal{O}(V_1) = k[x, x^{-1}] = k[x, y]/(xy - 1)$.
(2) Construct a morphism $V_2 \to \mathbb{A}^1_k$ whose image is V_1.
(3) Show that the local ring of V_2 at any point is isomorphic to R. Are V_2 and $\mathbb{A}^1_k$ isomorphic?
(4) Show that V_2 can be endowed with a structure of affine algebraic group.

Solution 2.

(1) We clearly have an inclusion $k[x, x^{-1}] \hookrightarrow \mathcal{O}(V_1)$, because any element $g(x)/x^n$ of $k[x, x^{-1}]$ is by definition a regular function on V_1. Let us show that every regular function is of this form.

Let $f\colon V_1 \to k$ be a regular function. Let $\{U_i\}_i$ be an open cover of V_1 and let $g_i, h_i \in k[x]$ be such that h_i is non-zero on U_i and $f|_{U_i} = g_i/h_i$ for all i. By applying Exercise 4.5 to $\mathbb{A}^1_k$, we in fact obtain $g_i/h_i = g_j/h_j$ for all i, j. As $k[x]$ is a UFD, every fraction has a unique representation in lowest terms, i.e. there exist $g, h \in k[x]$ coprime such that $g/h = g_i/h_i$ for all i. In particular, we have $h \mid h_i$ for all i, so that $V(h) \subseteq V(h_i)$, and thus

$$\mathbb{A}^1_k \setminus V(h) \supseteq \bigcup_i \mathbb{A}^1_k \setminus V(h_i) \underset{\substack{\uparrow \\ \boxed{h_i \text{ non-zero on } U_i}}}{\supseteq} \bigcup_i U_i = V_1.$$

Therefore $V(h) \subseteq \{0\}$. Up to scaling, we obtain that $h = x^n$ for some $n \in \mathbb{Z}_{\geq 0}$, and thus $f(x) = g(x)/x^n$ for all $x \in V_1$. Hence we obtain that the inclusion $k[x, x^{-1}] \hookrightarrow \mathcal{O}(V_1)$ is surjective, i.e. $\mathcal{O}(V_1) = k[x, x^{-1}]$.

To conclude, it suffices to show that $k[x, x^{-1}] \cong k[x, y]/(xy - 1)$. On the one hand, the morphism

$$\begin{gathered} k[x, y] \to k[x, x^{-1}] \\ x \mapsto x, y \mapsto x^{-1} \end{gathered}$$

has kernel $(xy-1)$ and thus we obtain an induced morphism $\varphi\colon k[x, y]/(xy-1) \to k[x, x^{-1}]$, sending $p(x, y) + (xy - 1)$ to $p(x, x^{-1})$. On the other hand, the composition

$$k[x] \hookrightarrow k[x, y] \to k[x, y]/(xy - 1)$$

sends x to $x + (xy - 1)$, which is a unit (with multiplicative inverse given by $y + (xy - 1)$). Hence by the universal property of localization, we obtain an induced morphism $\psi\colon k[x, x^{-1}] \to k[x, y]/(xy - 1)$, sending $g(x)/x^n$ to $g(x)y^n + (xy - 1)$. It is then straightforward to check that φ and ψ are mutually inverse.

(2) The projection $(v_1, v_2) \longmapsto v_1$ works. On structure rings, it is given by the morphism

$$\begin{gathered} k[x] \to k[x, y]/(xy - 1) \\ x \mapsto x + (xy - 1). \end{gathered}$$

(3) $k[x, x^{-1}]$ and $k[x]$ are not isomorphic so V_2 and $\mathbb{A}^1_k$ are not isomorphic.

For the question about local rings, consider a point $(a, b) \in V_2$. This corresponds to the maximal ideal $(\overline{x - a}, \overline{x - b})$ of $k[x, y]/(xy - 1)$. Under the isomorphism $k[x, y]/(xy - 1) \cong k[x, x^{-1}]$, this maximal ideal corresponds to the maximal ideal $(x - a) \subseteq k[x, x^{-1}]$, and thus the local ring $(k[x, y]/(xy - 1))_{(\overline{x-a}, \overline{x-b})}$ is isomorphic to $k[x, x^{-1}]_{(x-a)}$. By Exercise 4.1, it suffices to show that $k[x, x^{-1}]_{(x-a)} \cong k[x]_{(x-a)}$.

To do so, note that the localisation map $\iota\colon k[x] \to k[x, x^{-1}]$ satisfies $\iota^{-1}((x - a)) = (x - a)$, and thus we obtain an induced map on local rings $k[x]_{(x-a)} \to k[x, x^{-1}]_{(x-a)}$. It is straightforward to see that this is injective. To see surjectivity, let $(g/x^m)/(h/x^n)$ be an arbitrary element of the target. One then has $h(a) \neq 0$, and thus $(g \cdot x^n)/(h \cdot x^m)$ is a well-defined element of $k[x]_{(x-a)}$ mapping to $(g/x^m)/(h/x^n)$.

Although this might seem to be complicated algebra on first sight, geometrically it is really straightforward: the localization map $k[x] \to k[x, x^{-1}]$ corresponds to the open inclusion $\mathbb{A}^1_k \setminus \{0\} \hookrightarrow \mathbb{A}^1_k$. If $a \in \mathbb{A}^1_k \setminus \{0\}$ is any point, then the local ring at a should only depend on local information around a, i.e. it should be the same in all open neighborhoods of a. So as V_1 is an open neighborhood of a, the local ring of V_1 at a should be the same as the local ring of $\mathbb{A}^1_k$ at a.

Remark. The abstract algebraic fact which made the above proof work is the following: let R be a ring and let $S \subseteq R$ be a multiplicatively closed subset containing 1. Let also $T \subseteq S^{-1}R$ be a multiplicatively closed subset containing 1, and define

$$U \coloneqq \left\{ u \in R \mid \exists s \in S : \ \frac{u}{s} \in T \right\}.$$

Then we have an isomorphism $T^{-1}(S^{-1}R) \xrightarrow{\cong} U^{-1}R$ given by $(r/s)/(r'/s') \mapsto (rs')/(r's)$.

Applying this in the above situation to $R = k[x]$, $S = \{1, x, x^2, \ldots\}$ and $T = k[x, x^{-1}] \setminus (x-a)$ yields $U = k[x] \setminus (x-a)$ and $k[x, x^{-1}]_{(x-a)} \cong k[x]_{(x-a)}$.

(4) V_2 has a multiplication map given by

$$\begin{aligned} m \colon V_2 \times V_2 &\to V_2 \\ (a,b) \cdot (c,d) &\mapsto (ac, bd) \end{aligned}$$

which is well-defined as $(ac)(bc) = (ab)(cd) = 1$, and is clearly polynomial. The inverse map is

$$\begin{aligned} i \colon V_2 &\to V_2 \\ (a,b) &\mapsto (b,a) \end{aligned}$$

which is well-defined as $ba = ab = 1$ and also is polyonomial. Note that m is commutative and $m((a,b), i(a,b)) = (ab, ba) = (1,1)$, which is the neutral element for the multiplication m. In conclusion, this defines a structure of affine algebraic group on V_2.

Exercise 4.3. Let $V = V(y^2 - x^3)$. Let $\varphi : \mathbb{A}^1_k \to V$ be the morphism defined by $\varphi(t) = (t^2, t^3)$. From the first exercise, we call R the local ring of $\mathbb{A}^1_k$ at any point.

(1) Show φ is a bijective morphism, but is not an isomorphism.
(2) Let $P \in V$. Is the local ring of V at P isomorphic to R?

Solution 3. (1) φ is a bijection: there is a set-theoretic inverse $\psi \colon V \to \mathbf{A}^1_k$ given by $\psi(a,b) \coloneqq b/a$ on $V \setminus \{(0,0)\}$ and $\psi(0,0) \coloneqq 0$. However φ is not an isomorphism, because if it were, then ψ would have to be a morphism of affine algebraic varieties, but there doesn't exist any polynomial $p(x,y) \in k[x,y]$ such that $\psi(a,b) = p(a,b)$ for all $(a,b) \in V$. Indeed, if by contradiction p is such a polynomial, then the polynomial $q(x) = p(x^2, x^3) \in k[x]$ satisfies $q(t) = \psi(t^2, t^3) = t$ for all $t \in k$, and thus $q(x) = x$. But the coefficient of x in $p(x^2, x^3)$ is 0, contradiction.

We can also see it on the rings of functions, where φ is induced by the morphism

$$\begin{aligned} k[x,y] &\to k[x] \\ x &\mapsto x^2 \\ y &\mapsto x^3. \end{aligned}$$

The kernel is (y^2-x^3), but it is clearly not surjective, because x is not in the image. If φ was an isomorphism, the induced map $k[x,y]/(y^2-x^3) \to k[x]$ would have to be an isomorphism, contradiction.

(2) The problem of φ failing to be an isomorphism laid at $0 \in \mathbb{A}^1_k$, so let us try to show that actually $V_1 \cong V \setminus \{(0,0)\}$ (the precise definition of morphisms for quasi-affine algebraic varieties will come later, but it should be clear that the maps appearing should be morphisms for any reasonable definition). Indeed, $\psi|_{V\setminus\{(0,0)\}}$ maps $(a,b) \in V \setminus \{(0,0)\}$ to $b/a \in V_1$, which is a regular function on $V \setminus \{(0,0)\}$. Hence $\psi|_{V\setminus\{(0,0)\}} \colon V \setminus \{(0,0)\} \to V_1$ is a morphism of quasi-affine varieties, and it is inverse to $\varphi|_{V_1} \colon V_1 \to V \setminus \{(0,0)\}$. So we conclude that indeed $V_1 \cong V \setminus \{(0,0)\}$.

Although we haven't showed this yet, we will see later that the local ring remains the same under passing to an open subset (the proof of point (3) in Exercise 4.2 already showed this for the open set V_1 of $\mathbb{A}^1_k$). Therefore, the local ring of V at any point $P \in V \setminus \{(0,0)\}$ is isomorphic to R.

Let us show that this is not the case at $P = (0,0)$. For an element $p \in k[x,y]$, denote by $\overline{p}$ its class inside $k[x,y]/(y^2-x^3) = \Gamma(V)$. The ideal of $(0,0)$ inside $\Gamma(V)$ is $(\overline{x}, \overline{y})$, so the local ring $\mathcal{O}_P(V)$ is given by

$$\mathcal{O}_P(V) = \left(k[x,y] \Big/ (y^2-x^3) \right)_{(\overline{x},\overline{y})}.$$

Denote by $\mathfrak{m} = (\overline{x}/1, \overline{y}/1)$ its maximal ideal. To show that $\mathcal{O}_P(V)$ is not isomorphic to $R = k[x]_{(x)}$, we are going to show that $\mathfrak{m}$ is not principal (which is enough, as the maximal ideal of R is principal). So assume by contradiction that $\mathfrak{m} = (\overline{p}/\overline{q})$ for some $p, q \in k[x,y]$. As $\overline{x}/1, \overline{y}/1 \in \mathfrak{m}$, there exist $a,b,c,d \in k[x,y]$ with $b(0,0) \neq 0 \neq d(0,0)$ such that

$$\begin{aligned} \frac{\overline{a}}{\overline{b}}\frac{\overline{p}}{\overline{q}} &= \frac{\overline{x}}{1} \\ \frac{\overline{c}}{\overline{d}}\frac{\overline{p}}{\overline{q}} &= \frac{\overline{y}}{1}. \end{aligned}$$

If by contradiction $a(0,0) \neq 0$, then $\overline{a} \notin (\overline{x}, \overline{y})$, so $\overline{a}$ is a unit in $\mathcal{O}_P(V)$. Therefore, we obtain that $\overline{p}/\overline{q}$ is a multiple of $\overline{x}/1$, which gives $\mathfrak{m} = (\overline{x}/1)$. So in this case we may assume $\overline{p}/\overline{q} = \overline{x}/1$. But then the second equation above gives that

$$\overline{dy - cx} = 0$$

and thus $dy - cx \in (y^2 - x^3)$. Therefore, the polynomial $d(t^2,t^3)t^3 - c(t^2,t^3)t^2 \in k[t]$ is 0. Hence we obtain $d(t^2,t^3)t = c(t^2,t^3)$ so that in particular $c(0,0) = 0$, i.e. c has no constant term. But then $c(t^2,t^3)$ is divisible by t^2, i.e we can write $c(t^2,t^3) = t^2\widetilde{c}(t)$ for some $\widetilde{c}(t) \in k[t]$. But then we obtain $d(t^2, t^3) = t\widetilde{c}(t)$ and so $d(0,0) = 0$, which contradicts the fact that $\overline{d} \notin (\overline{x}, \overline{y})$.

Therefore, we may assume that $a(0,0) = 0$. From the above equations, we obtain that

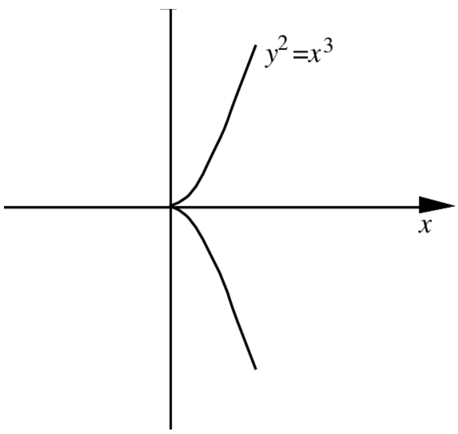
$$\overline{ap - xbq} = 0$$

and thus we have

$$a(t^2, t^3)p(t^2, t^3) - t^2 b(t^2, t^3)q(t^2, t^3) = 0.$$

As neither a nor p have a constant term, we obtain that $a(t^2, t^3)p(t^2, t^3)$ is divisible by t^4. But then $b(t^2, t^3)q(t^2, t^3)$ is divisible by t^2, so in particular we obtain $b(0,0)q(0,0) = 0$. This contradicts the fact that $\overline{b}, \overline{q} \notin (\overline{x}, \overline{y})$. In conclusion, the maximal ideal of $\mathcal{O}_P(V)$ cannot be principal, while the maximal ideal of R is, so they can't be isomorphic.

Remark. The geometric reason why $\mathcal{O}_P(V)$ is not isomorphic to the local rings of $\mathbb{A}^1_k$, is that the curve $y^2 - x^3$ has a singularity at $P = (0,0)$, i.e. it is not smooth there. In fact, it has some sort of 'sharp corner' at P, as the following picture suggests:



Algebraically, the way one defines what it means for some variety V to be singular/smooth at a point $P \in V$ is through its tangent space. The algebraic analogue of the tangent space is (the dual of) the k-vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$; this is called the *Zariski tangent space*. If the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ is the same as the dimension of V, then we say that V is smooth at P (intuitively, it means that V looks like $\mathbb{A}_k^{\dim V}$ if we 'zoom in close enough'). If on the other hand we have $\dim_k \mathfrak{m}_P/\mathfrak{m}_P^2 > \dim V$, we say that V is singular at P. Furthermore, by Nakayama's lemma, the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k-vector space is in fact the same as the minimal number of generators of the ideal $\mathfrak{m}_P$ inside $\mathcal{O}_P(V)$. What we showed above is that for $V = V(y^2 - x^3)$ and $P = (0,0)$, the minimal number of generators of $\mathfrak{m}_P$ inside $\mathcal{O}_P(V)$ is at least 2, so

$$\dim_k \mathfrak{m}_P/\mathfrak{m}_P^2 > 1 = \dim V.$$

Hence V has a singularity at P, whereas $\mathbb{A}^1_k$ is smooth at every point.

Exercise 4.4. Let $V = V(Y^2 - X^2(X+1))$ and x, y the residues of X, Y in $\Gamma(V)$. Let $z = \frac{y}{x} \in k(V)$. Find the poles of z and z^2.

Solution 4. Note that

$$z = \frac{y}{x} = \frac{x(x+1)}{y},$$

so the only possible pole is where both x and y are 0, i.e. at $(0,0)$. For $p \in k[X,Y]$, denote by $\overline{p}$ its class in $\Gamma(V)$. Assume by contradiction that we can write $z = \overline{p}/\overline{q}$ with $q(0,0) \neq 0$. Then we have

$$\overline{Yq - Xp} = 0,$$

or equivalently, there exists $r \in k[X,Y]$ such that

$$Yq(X,Y) - Xp(X,Y) = r(X,Y)(Y^2 - X^2(X+1)).$$

Plugging in $X = 0$ gives

$$Yq(0,Y) = Y^2 r(0,Y),$$

so we obtain $q(0,0) = 0$, contradiction. Hence z has a pole at $(0,0)$.

On the other hand, we have $z^2 = \frac{y^2}{x^2} = \frac{x^2(x+1)}{x^2} = x+1$. As this has no denominator, z^2 has no poles.

Exercise 4.5.

(1) Prove Corollary 2.9 fom class: Let V be a quasi-affine variety and $f, g \in \mathcal{O}(V)$ two regular functions, such that $f_{|U} = g_{|U}$ for some non-empty open $U \subset V$. Then $f = g$.
(2) Let V be an affine variety and $f \in k(V)$ a rational function. Show that f defines a continous function $U \to k$, for some non empty open subset $U \subset V$. Furthermore f is uniquely determined by this function.

Solution 5.

(1) Consider $h = f - g \in \mathcal{O}(V)$. As h is continuous, we have that $h^{-1}(\{0\}) \subseteq V$ is closed, but it also contains U. As V is irreducible, U is dense, and therefore we must have $h^{-1}(0) = V$. That is, we have $h = 0$, and thus $f = g$.
(2) V is irreducible so $\Gamma(V)$ is integral and we can write f as g/h with $g, h \in \Gamma(V)$. Then the zero set of h is a closed subset of V and we can take U to be its complement. The only Zariski closed subsets of k are $\varnothing, k$ and finite sets of points. Checking the continuity on singletons is enough. Using translations, it suffices to check at 0. Now $f^{-1}(0)$ is Zariski closed since $f^{-1}(0) = V(g) \cap U$ is Zariski closed.

    Using projective space: We can see f as a function $V \to \mathbb{P}^1$. Then, $f^{-1}(\infty)$ is closed and its complement is the open subset U.

    Let $f, g \in k(V)$ and write $f = a/b$ and $g = c/d$ for some $a, b, c, d \in \Gamma(V)$. Assume that f, g define continuous functions $F\colon U_1 \to k$ resp. $G\colon U_2 \to k$, and that there exists a non-empty open subsets $W \subseteq U_1 \cap U_2$ such that $F|_W = G|_W$. By further shrinking W, we may suppose that b, d are non-zero on W. By construction and using point (1), we have $F(x) = a(x)/b(x)$

and $G(x) = c(x)/d(x)$ for all $x \in W$. As $F|_W = G|_W$, we obtain that $(ad)|_W = (bc)|_W$. By point (1), this gives $ad = bc$, so we obtain $f = g$.

Exercise 4.6. * Let $F \in k[x, y]$ be an irreducible polynomial of degree at most 2. Show that $V(F)$ is either isomorphic to $V_1 = \mathbb{A}^1_k$ or $V_2 = V(xy - 1)$. Specify in which case it is isomorphic to V_1 (resp. V_2). (Hint: Use linear changes of coordinates to eliminate monomials in F)

Solution 6. A degree 1 irreducible polynomial is of the form $F = ax + by + c$ with a or $b \neq 0$. Assume $a \neq 0$. Then we have the following surjective morphism

$$\begin{aligned} k[x,y] &\to k[x] \\ x &\mapsto -a^{-1}(bx + c) \\ y &\mapsto x \end{aligned}$$

whose kernel is (F). Thus $V(F)$ is isomorphic to $\mathbb{A}^1_k$.

Now suppose F is an irreducible polynomial of degree 2 in $k[x, y]$. We can write

$$F(x, y) = ax^2 + by^2 + cxy + dx + ey + f = 0$$

- if $a = 0$ and $b = 0$, then $c \neq 0$. Using
$$cxy + dx + ey = c(x + \frac{e}{c})(y + \frac{d}{c}) - ed$$
we get $F = cXY + f'$ with $X = x + \frac{e}{c}$, $Y = y + \frac{d}{c}$ and $f' = f - ed$. Then F irreducible implies $f' \neq 0$. If we write $X' = \frac{c}{f'}$, then $F = f'XY - f'$. It is then clear that $V(F) = V(XY - 1)$.
  Note that these affine changes of variables are admitted because they induce isomorphism of rings.
- Up to changing x and y, we may assume $a \neq 0$. Then writing
$$F(x, y) = ax^2 + (cy + d)x + by^2 + ey + f,$$
we may complete the square and replace $X = \sqrt{a}(x + a^{-1}(cy + d)/2)$ to obtain $F(X, y) = X^2 + b'y^2 + e'y + f'$.
  - If $b' = 0$ we must have $e' \neq 0$, otherwise F would be reducible. Hence we can replace $Y = e'y + f'$ and obtain $F(X, Y) = X^2 + Y$. But then use the isomorphism
  $$\begin{aligned} k[X,Y]/(X^2 + Y) &\to k[X] \\ X &\mapsto X \\ Y &\mapsto -X^2 \end{aligned}$$
  to conclude that $V(F) \cong \mathbb{A}^1_k$.
  - If $b' \neq 0$, we can again complete the square and assume $e' = 0$. As $X^2 + b'y^2$ is reducible over an algebraically closed field (we can write $X^2 + b'y^2 = (X + i\sqrt{'b}y)(X - i\sqrt{b'}y))$, we then must have $f' \neq 0$. Up to scaling X and y we may assume that $e' = f' = -1$, so we are left with $F(X, y) = X^2 - y^2 - 1$. Factoring $X^2 + y^2 = (X + y)(X - y)$ and replacing $u = X + y$ and $v = X - y$, we obtain $F(u, v) = uv - 1$. Therefore, we obtain again $V(F) \cong V_2$.